A Nonlocal Formulation of Rotational Water Waves

A.C.L Ashton* and A.S Fokas†

Department of Applied Mathematics
and Theoretical Physics,
University of Cambridge,
CB3 0WA, UK.

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Abstract

The classical equations of irrotational water waves have recently been reformulated as a system of two equations, one of which is an explicit non-local equation for the wave height and for the velocity potential evaluated on the free surface. Here we first extend this formalism to n-dimensions, $n > 2$, and then derive rigorously the linear limit of these equations. Furthermore, for $n = 2$, we generalise the relevant forumlation to the case of constant vorticity and to the case where the free surface is described by a multi-valued function. Also, in the two dimensional case we derive a sequence of Hamiltonian systems, hence providing an approximation in the asymptotic limit of certain physical small parameters.

*a.c.l.ashton@damtp.cam.ac.uk
†t.fokas@damtp.cam.ac.uk

1 Introduction

We consider the classical problem in hydrodynamics concerning the propagation of surface waves generated by an incompressible fluid with free surface. In the case of irrotational flow we consider the problem in n spatial dimensions, whereas in the rotational case we confine attention to $n = 2$. We consider the problem in its full generality so that the depth of the fluid may not be constant, and consider *all* solutions, not only those that arise under the travelling wave assumption. In addition, we include the effect of surface tension so that capillary-type waves are included in our study.

We denote by $\mathcal{B}_h$ the bottom surface:

$$\mathcal{B}_h = \{(x, y) : x \in \mathbf{R}^{n-1},\ y = -h_0 + h(x)\}, \tag{1.1}$$

where h_0 is constant and $h(x)$ is a real valued function; the notation $\mathcal{B}_\infty$ will denote infinite depth. We denote by $\mathcal{S}_\eta$ the free surface:

$$\mathcal{S}_\eta = \{(x, y) : x \in \mathbf{R}^{n-1},\ y = \eta(x, t)\} \qquad \text{for } t \geq 0. \tag{1.2}$$

We refer to η as the height of the wave and we assume $\eta + h_0 > h$ for each $x \in \mathbf{R}^{n-1}$. The domain of the problem is the region between $\mathcal{B}_h$ and $\mathcal{S}_\eta$, which is denoted by Ω.

1.1 The Irrotational Case in n Dimensions

In the irrotational case we introduce the velocity potential ϕ, where $u = \nabla\phi$, and then the governing equations become:

$$\Delta\phi = 0 \qquad \text{in } \Omega, \tag{1.3a}$$

$$\nabla\phi \cdot N_{\mathcal{B}} = 0 \qquad \text{on } \mathcal{B}_h, \tag{1.3b}$$

$$\nabla\phi \cdot N_{\mathcal{S}} = \eta_t \qquad \text{on } \mathcal{S}_\eta, \tag{1.3c}$$

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + g\eta = f(\eta) \qquad \text{on } \mathcal{S}_\eta, \tag{1.3d}$$

where g is the acceleration due to gravity, ∇ denotes the usual vector gradient, $N_{\mathcal{B}}$ is the exterior normal to $\mathcal{B}_h$ and $N_{\mathcal{S}}$ is the exterior normal to $\mathcal{S}_\eta$, i.e.

$$\nabla = (\partial_x, \partial_y), \qquad N_{\mathcal{B}} = (-\partial_x h, -1), \qquad N_{\mathcal{S}} = (-\partial_x \eta, 1). \tag{1.4a}$$

The right hand side of (1.3d) is functionally dependent on η through:

$$f(\eta) = \frac{\sigma}{\rho}\mathrm{Div}\left(\frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}}\right) \tag{1.4b}$$

which is a measure of the effect of surface tension (σ and ρ denote the constant surface tension and density respectively). Equation (1.3a) is a consequence of

incompressibility, (1.3b) is the Neumann condition satisfied on $\mathcal{B}_h$ and (1.3c), (1.3d) are the kinematic, Bernoulli conditions respectively on $\mathcal{S}_\eta$.

We will reformulate the problem in terms of the functions η and q where $q(x,t) = \phi(x,\eta,t)$, i.e. q represents the potential on the free surface $\mathcal{S}_\eta$. It was shown by Zakharov in his classical paper (see [17]) that in the case $n = 2$ the pair (η, q) constitutes a canonically conjugate pair in the Hamiltonian formulation of this problem. The generalisation of Zakarov's result to n-dimensions is straightforward. Indeed, in the case of constant $\mathcal{B}_h$ and zero surface tension equations (1.3) admit a Hamiltonian formulation with respect to the Hamilontian

$$H = \iint_\Omega \tfrac{1}{2}|\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}y + \int \tfrac{1}{2}g\eta^2 \,\mathrm{d}x \tag{1.5}$$

and the standard symplectic structure. Here $\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1}$ is the Lebesgue measure on $\mathbf{R}^{n-1}$.

Applying the chain rule to the expression $q(x,t) = \phi(x,\eta,t)$ we find the following relations:

$$\partial_x q = \partial_x\phi + (\partial_y\phi)\partial_x\eta, \tag{1.6a}$$

$$q_t = \phi_t + \phi_t\eta_t. \tag{1.6b}$$

It is possible to solve for $\nabla\phi$ in terms of the q and η following a similar calculation with that of [1]. Indeed, using (1.3c) in equations (1.6a) we find the following nonsingular set of equations for $\partial_x\phi$:

$$\partial_x q - \eta_t\partial_x\eta = (\mathbf{I} + \partial_x\eta \otimes \partial_x\eta)\cdot\partial_x\phi \tag{1.7}$$

and then equation (1.3c) gives ϕ_y in terms of q and η. In the new variables the dynamic boundary condition on $\mathcal{S}_\eta$, i.e. (1.3d), becomes

$$q_t + \tfrac{1}{2}|\partial_x q|^2 + g\eta - \frac{(\eta_t + \partial_x\eta\cdot\partial_x q)^2}{2(1+|\partial_x\eta|^2)} = f(\eta). \tag{1.8}$$

In what follows, the notation $\int \mathrm{d}x$ will be used to denote an integral over $\mathbf{R}^{n-1}$.

1.2 The Rotational Case in Two Dimensions

In this section we derive the equations for the free surface problem for a two dimensional fluid with constant vorticity. Denoting the velocity of the flow by (u,v), the Euler equations for invisid flow are the following equations:

$$u_t + uu_x + vu_y = -P_x, \tag{1.9a}$$

$$v_t + uv_x + vv_y = -P_y - g, \tag{1.9b}$$

$$u_x + v_y = 0. \tag{1.9c}$$

Let ω denote the vorticity of the fluid, i.e.

$$\omega = v_x - u_y. \tag{1.10}$$

Eliminating P from (1.9a) and (1.9b) we find:

$$\frac{\partial \omega}{\partial t} + (u\partial_x + v\partial_y)\omega = 0. \tag{1.11}$$

We restrict attention to the case in which $\omega = \gamma$ is constant throughout Ω, so that (1.11) is satisfied identically. The domain Ω is simply connected, therefore we can introduce a globally defined stream function $\psi(t, x, y)$ so that:

$$u = \psi_y, \qquad v = -\psi_x, \qquad (x, y) \in \Omega. \tag{1.12}$$

Replacing in equation (1.10) u by ψ_y and v by $-\psi_x$ we find $\Delta\psi = -\gamma$. Thus, the function defined by $\psi^h = \psi + \frac{1}{2}y^2\gamma$, is harmonic in Ω. In fact, ψ^h is only defined upto the addition of an arbitrary function of time, so by abuse of notation we writeg ψ^h to mean the $[\psi^h]$, i.e. the equivalence class of such functions that differ only by a function of time. Let φ to be the harmonic conjugate of ψ^h, i.e.

$$\varphi_x = \psi_y + \omega y, \tag{1.13a}$$
$$\varphi_y = -\psi_x. \tag{1.13b}$$

The function φ is harmonic throughout Ω.

Adding and subtracting in equations (1.9a) and (1.9b) the terms vv_x and uu_y respectively, the equations become

$$u_t + \tfrac{1}{2}\partial_x(u^2 + v^2) - \gamma v = -P_x, \tag{1.14a}$$
$$v_t + \tfrac{1}{2}\partial_y(u^2 + v^2) + \gamma u = -P_y - g. \tag{1.14b}$$

Using in equations (1.14) the identities

$$u_t = (\psi_y)_t = (\varphi_x)_t, \quad v = -\psi_x, \quad v_t = -(\psi_x)_t = (\varphi_y)_t, \quad u = \psi_y,$$

equations (1.14) can be integrated to give the following equation:

$$\varphi_t + \tfrac{1}{2}|\nabla\psi|^2 + \gamma\psi + P + gy = \alpha(t), \qquad (x, y) \in \Omega, \tag{1.15}$$

where $\alpha(t)$ is some function of time. Since we are still dealing with an equivalence class of functions ψ and φ we can absorb function of time into them, so we arrive at

$$\varphi_t + \tfrac{1}{2}|\nabla\psi|^2 + \gamma\psi + gy = P_{\text{atm}} - P, \qquad (x, y) \in \Omega, \tag{1.16}$$

where $P_{\rm atm}$ is the atmospheric pressure above the free surface $\mathcal{S}_\eta$. On the free surface $\mathcal{S}_\eta$ we have the dynamic boundary condition

$$P_{\rm atm} - P = f(\eta), \qquad f(\eta) \stackrel{\rm def}{=} \sigma \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)_x . \tag{1.17}$$

The streamfunction ψ is *not* constant on streamlines in the case of constant vorticity, however since we are dealing with an equivalence class of functions we may absorb this function of time into ψ and use the normalisation $\psi = 0$ on $\mathcal{S}_\eta$. Hence (1.16) and (1.17) imply the following nonlinear boundary condition:

$$\partial_t \varphi + \tfrac{1}{2}|\nabla\psi|^2 + g\eta = f(\eta) \qquad \text{on } \mathcal{S}_\eta. \tag{1.18}$$

The kinematic condition on $\mathcal{S}_\eta$ and the condition on $\mathcal{B}_h$ are the same as in the irrotational case.

In summary, if the vorticity equals the constant γ, then the functions $\varphi(x,y,t)$ and $\eta(x,t)$ satisfy the following boundary value problem:

$$\Delta\varphi = 0 \qquad \text{in } \Omega, \tag{1.19a}$$
$$(\varphi_x - \gamma y, \varphi_y)\cdot N_{\mathcal{B}} = 0 \qquad \text{on } \mathcal{B}_h, \tag{1.19b}$$
$$(\varphi_x - \gamma y, \varphi_y)\cdot N_{\mathcal{S}} = \eta_t \qquad \text{on } \mathcal{S}_\eta, \tag{1.19c}$$
$$\varphi_t + \tfrac{1}{2}\varphi_y^2 + \tfrac{1}{2}(\varphi_x - \gamma y)^2 + gy = f(\eta) \qquad \text{on } \mathcal{S}_\eta, \tag{1.19d}$$

where $N_{\mathcal{B}}$, $N_{\mathcal{S}}$ and $f(\eta)$ are defined in (1.4).

As in the irrotational case, we will formulate the problem in terms of η and ξ where $\xi(x,t) = \varphi(x,\eta,t)$, i.e ξ represemts the pseudo-potential φ on the free surface. Computing ξ_x using the chain rule (compare with (1.6a)) and invoking (1.19c) we find the following pair of equations for (φ_x, φ_y) on $\mathcal{S}_\eta$:

$$\xi_x = \varphi_x + \eta_x\varphi_y,$$
$$\eta_t - \gamma\eta\eta_x = \varphi_y - \eta_x\varphi_x.$$

Solving these equations for (φ_x, φ_y) we find the following relations on $\mathcal{S}_\eta$:

$$(1+\eta_x^2)\varphi_x = \xi_x - \eta_t\eta_x + \gamma\eta\eta_x^2, \tag{1.20a}$$
$$(1+\eta_x^2)\varphi_y = \eta_t + \eta_x\xi_x - \gamma\eta\eta_x. \tag{1.20b}$$

Furthermore, using (1.20) in the boundary condition (1.19d) on $\mathcal{S}_\eta$ we find:

$$\xi_t + \tfrac{1}{2}\xi_x^2 + g\eta + \frac{\gamma\eta(2\eta_t\eta_x - 2\xi_x + \gamma\eta)}{2(1+\eta_x^2)} - \frac{(\eta_t + \eta_x\xi_x)^2}{2(1+\eta_x^2)} = f(\eta). \tag{1.21}$$

It is reassuring to note that (1.21) reduces to (1.8) when $\gamma = 0$.

1.3 Multivalued Free surface

In the above analysis we have assumed that $\eta(x,t)$ is a single valued function. However, in general, this assumption can be violated. In particular, it is known that in the case of water waves with constant vorticity, $\eta(x,t)$ quickly becomes a multi-valued function, as observed by [14]. In what follows, we show that it is conceptually straightforward to modify our analysis so that it can address the case of a multi-valued free surface.

The free surface is assumed to be a one-dimensional C^2-differentiable manifold, so features such as cusps and self-intersections are ignored. We set:

$$\mathcal{S} = \{(x,y) \in \mathbf{R}^2 : x = X(\lambda,t), y = Y(\lambda,t)\}, \qquad t \geq 0$$

where $\lambda \in \mathbf{R}$ is the arc-length along the curve and $X(\lambda,\cdot)$, $Y(\lambda,\cdot)$ are assumed to be C^2. The outward normal is now given by $N_{\mathcal{S}} = (-\dot{Y},\dot{X})$, where the dot denotes differentiation with respect to λ. The kinematic boundary condition in (1.19c) now becomes:

$$(\varphi_x - \gamma y - X_t, \varphi_y - Y_t) \cdot N_{\mathcal{S}} = 0, \qquad \text{on } \mathcal{S}. \tag{1.22}$$

Again, we introduce a function ξ to represent the pseudo-potential on the free surface, i.e. $\xi(\lambda,t) = \varphi(X(\lambda,t), Y(\lambda,t), t)$. In analogy with equations (1.6), we note the following relations on $\mathcal{S}$:

$$\dot{\xi} = \dot{X}\varphi_x + \dot{Y}\varphi_y, \tag{1.23a}$$

$$\xi_t = X_t\varphi_x + Y_t\varphi_y + \varphi_t. \tag{1.23b}$$

Equations (1.22) and (1.23) constitute a non-singular system for the functions $\{\varphi_x, \varphi_y, \varphi_t\}$ on the free surface $\mathcal{S}$, so we can invert these relationships to express these functions in terms of $\{X, Y, \eta\}$. After some algebra, one finds the following relations on $\mathcal{S}$:

$$\varphi_x = \frac{\dot{\xi}\dot{X} + \dot{Y}(\dot{Y}(\gamma Y + X_t) - \dot{X}Y_t)}{(\dot{X}^2 + \dot{Y}^2)} \tag{1.24a}$$

$$\varphi_y = \frac{\dot{\xi}\dot{Y} - \dot{X}(\dot{Y}(\gamma Y + X_t) - \dot{X}Y_t)}{(\dot{X}^2 + \dot{Y}^2)} \tag{1.24b}$$

$$\varphi_t = \xi_t - \frac{\dot{\xi}(\dot{X}X_t + \dot{Y}Y_t) + (X_t\dot{Y} - Y_t\dot{X})(\dot{Y}(\gamma Y + X_t) - \dot{X}Y_t)}{(\dot{X}^2 + \dot{Y}^2)} \tag{1.24c}$$

Using these expressions on $\mathcal{S}$, the left hand side of the Bernoulli condition in (1.19d) becomes:

$$\xi_t + gY - \tfrac{1}{2}X_t^2 - \tfrac{1}{2}Y_t^2 + \frac{\gamma Y(2\dot{X}\dot{Y}Y_t - 2\dot{X}^2X_t - 2\dot{\xi}\dot{X} + \gamma\dot{X}^2Y)}{2(\dot{X}^2 + \dot{Y}^2)} + \frac{(\dot{\xi} - \dot{X}X_t - \dot{Y}Y_t)^2}{2(\dot{X}^2 + \dot{Y}^2)}$$

and the term due to surface tension is:

$$\sigma \frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2 + \dot{Y}^2)^{3/2}}.$$

Here the coefficient of σ is the intrinsic curvature of the surface $\mathcal{S}$. Combining these expressions gives the Bernoulli condition on $\mathcal{S}$ in terms of the surface parameters (X, Y) and the potential on the free surface, $\xi(\lambda, t)$:

$$\xi_t + gY - \tfrac{1}{2}X_t^2 - \tfrac{1}{2}Y_t^2 + \frac{\gamma Y(2\dot{X}\dot{Y}Y_t - 2\dot{X}^2 X_t - 2\dot{\xi}\dot{X} + \gamma \dot{X}^2 Y)}{2(\dot{X}^2 + \dot{Y}^2)} + \frac{(\dot{\xi} - \dot{X}X_t - \dot{Y}Y_t)^2}{2(\dot{X}^2 + \dot{Y}^2)} - \sigma \frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2 + \dot{Y}^2)^{3/2}} = 0. \quad (1.25)$$

We note that in the case $X = \lambda$, $Y = Y(X, t)$ this equation reduces to (1.21).

The above analysis shows that although it is straightforward to incorperate the case of a multivalued free surface, the relevant formulae become more complicated, so for convenience we will present most of our results assuming that $\eta(x, t)$ is single valued.

2 The Non-Local Formulation

A novel non-local formulation in two and three dimensions for irrotational water waves was presented by [1]. Here we extend this formulation to rotational water waves with constant vorticity in two dimensions and for irrotational water waves in an arbitrary number of dimensions.

The formulation by [1] is based on the existence of the so-called *global relation* (see [9]). The global relation is a consequence of the following fact: suppose that the functions u and v are harmonic in $\Omega \subset \mathbf{R}^{n-1} \times \mathbf{R}$. Then,

$$\mathrm{Div}\,\{(\partial_y u)\partial_x v + (\partial_y v)\partial_x u\} + \partial_y\,\{(\partial_y u)(\partial_y v) - \partial_x u \cdot \partial_x v\} = 0 \qquad (2.1)$$

for each $(x, y) \in \Omega$. This can be verified by expanding out the left hand side of (2.1) to find:

$$(\partial_y v)\Delta u + (\partial_y u)\Delta v,$$

which vanishes in Ω since both u and v are harmonic in Ω.

2.1 Irrotational n-dimensional Case

We assume $\phi(x, y, t)$ and $\eta(x, t)$ have sufficient decay as $|x| \to \infty$ for each (y, t) in order for the integrals that follow to exist. This is indeed the case, if for example $\nabla\phi \in L^2(\Omega)$ and $\eta, \eta_t \in L^2(\mathbf{R}^{n-1})$; the justifications of these assumptions is beyond the scope of this present work.

Suppose $k \in \mathbf{R}^{n-1}$ and define $v \in C^\infty(\Omega)$ by:

$$v(x,y) = \exp(\mathrm{i}k \cdot x + \kappa y), \qquad \kappa = \pm\sqrt{k_1^2 + \cdots k_{n-1}^2}.$$

Then v is harmonic in Ω and the following holds for $(x,y) \in \Omega$:

$$\mathrm{Div}\left\{v(\mathrm{i}k\partial_y\phi + \kappa\partial_x\phi)\right\} + \partial_y\left\{v(\kappa\partial_y\phi - \mathrm{i}k \cdot \partial_x\phi)\right\} = 0$$

Integrating this expression over Ω and applying the divergence theorem gives:

$$\int_{\mathcal{S}_\eta} e^{\mathrm{i}k\cdot x+\kappa y}\left(\mathrm{i}k\partial_y\phi + \kappa\partial_x\phi, \kappa\partial_y\phi - \mathrm{i}k \cdot \partial_x\phi\right) \cdot N_{\mathcal{S}}\,\mathrm{d}x \\ + \int_{\mathcal{B}_h} e^{\mathrm{i}k\cdot x+\kappa y}\left(\mathrm{i}k\partial_y\phi + \kappa\partial_x\phi, \kappa\partial_y\phi - \mathrm{i}k \cdot \partial_x\phi\right) \cdot N_{\mathcal{B}}\,\mathrm{d}x = 0 \quad (2.2)$$

We have discarded the contribtions from $\partial\mathbf{R}^{n-1}$ using our assumption about the decay of the fields. The contribution from $\mathcal{S}_\eta$ in (2.2), is given by the following integral:

$$\int_{\mathcal{S}_\eta} e^{\mathrm{i}k\cdot x+\kappa y}\big[\kappa\left(\partial_y\phi - \partial_x\phi \cdot \partial_x\eta\right) - \mathrm{i}k \cdot \left(\partial_x\phi + (\partial_y\phi)\partial_x\eta\right)\big]\,\mathrm{d}x.$$

From (1.3c) the first term in the above integrand becomes η_t, and by (1.6), the second term becomes $\partial_x q$. This gives the expression

$$\int e^{\mathrm{i}k\cdot x+\kappa\eta}\left(\kappa\eta_t - \mathrm{i}k \cdot \partial_x q\right)\,\mathrm{d}x. \quad (2.3)$$

Now we consider the contribution from $\mathcal{B}_h$ in (2.2). We introduce $Q(x,t) = \phi(x, -h_0 - h, t)$, which represents the potential on the bottom $\mathcal{B}_h$. In analogy with (1.6), an application of the chain rule yields:

$$\partial_x Q = \partial_x\phi - (\partial_y\phi)\partial_x h. \quad (2.4)$$

Recalling that $N_{\mathcal{B}} = (-\partial_x h, -1)$, the contribution from $\mathcal{B}_h$ in (2.2) gives

$$-\int_{\mathcal{B}_h} e^{\mathrm{i}k\cdot x+\kappa y}\big[\kappa\left(\partial_y\phi + \partial_x\phi \cdot \partial_x h\right) - \mathrm{i}k \cdot \left(\partial_x\phi - (\partial_y\phi)\partial_x h\right)\big]\,\mathrm{d}x. \quad (2.5)$$

Equation(1.3b) implies that $\nabla\phi \cdot N_{\mathcal{B}} = 0$ on $\mathcal{B}_h$, so the first term in the above integrand vanishes. Using the result in (2.4) we find that the expression in (2.5) becomes:

$$\int e^{\mathrm{i}k\cdot x-\kappa(h_0+h)}(\mathrm{i}k \cdot \partial_x Q)\,\mathrm{d}x. \quad (2.6)$$

Now combining the results in (2.3), (2.6) and (2.2) we find that equation (2.2) reduces to the following equation:

$$\int e^{\mathrm{i}k\cdot x+\kappa\eta}\left(\kappa\eta_t - \mathrm{i}k \cdot \partial_x q\right)\,\mathrm{d}x + \int e^{\mathrm{i}k\cdot x-\kappa(h_0+h)}(\mathrm{i}k \cdot \partial_x Q)\,\mathrm{d}x = 0. \quad (2.7)$$

By evaluating (2.7) at $\pm|\kappa|$ and adding/subtracting the resulting equations, we arrive at the following result:

Proposition 1 (Irrotational Waves in n-Dimensions)**.** *The boundary value problem given in* (1.3a)-(1.3c) *is equivalent to the pair of integro-differential equations:*

$$\int e^{\mathrm{i}k\cdot x}\left[\kappa\eta_t\sinh(\kappa\eta) - \mathrm{i}k\cdot\partial_x q\cosh(\kappa\eta) + \mathrm{i}k\cdot\partial_x Q\cosh(\kappa(h+h_0))\right]\mathrm{d}x = 0 \quad (2.8)$$

and

$$\int e^{\mathrm{i}k\cdot x}\left[\kappa\eta_t\cosh(\kappa\eta) - \mathrm{i}k\cdot\partial_x q\sinh(\kappa\eta) - \mathrm{i}k\cdot\partial_x Q\sinh(\kappa(h+h_0))\right]\mathrm{d}x = 0 \quad (2.9)$$

valid for each $k\in\mathbf{R}^{n-1}$*. These equations, together with the Bernoulli condition* (1.8) *constitute three equations for the three unknowns* (η, q, Q).

2.2 The Rotational two-dimensional Case

Let $n=2$ and also confine attention to the case in which $\mathcal{B}_h=\mathcal{B}_0$ is constant. Using a similar approach to the n-dimensional, irrotational case, we find the non-local integro-differential equation for the pseudo-potential $\varphi(x,t)$ and the wave height $\eta(x,t)$:

$$\int_{\mathcal{S}_\eta} e^{\mathrm{i}kx\pm ky}\left[\mathrm{i}(\varphi_x+\varphi_y\eta_x)\pm(\eta_x\varphi_x-\varphi_y)\right]\mathrm{d}x + \int_{\mathcal{B}_0} e^{\mathrm{i}kx\pm ky}\left[\mathrm{i}\varphi_x\mp\varphi_y\right]\mathrm{d}x = 0,$$

which is valid for $k\in\mathbf{R}$. Invoking the boundary conditions (1.19b), (1.19c) and (1.20) this expression becomes

$$\int e^{\mathrm{i}kx\pm k\eta}\left[\mathrm{i}\xi_x\pm(\gamma\eta\eta_x-\eta_t)\right]\mathrm{d}x + \int e^{\mathrm{i}kx\mp h_0}\mathrm{i}\varphi_x(x,-h_0,t)\,\mathrm{d}x = 0.$$

Subtracting the above two expressions eliminates the $\mathcal{B}_0$ integral completely and we are left with the global relation for two dimensional water waves with constant vorticity γ:

$$\int e^{\mathrm{i}kx}\left[\xi_x\sinh\left(k(\eta+h)\right) + \mathrm{i}(\eta_t-\gamma\eta\eta_x)\cosh(k(\eta+h))\right]\mathrm{d}x = 0,$$

where we have dropped the subscript on h. This leads us to the following proposition.

Proposition 2 (Two Dimensional Water Waves with Constant Vorticity)**.** *The boundary value problem in* (1.19a)-(1.19d) *for* (φ,η) *is equivalent to the following pair of integro-differential equations for* (ξ,η)*:*

$$\int e^{\mathrm{i}kx}\left[\xi_x\sinh\left(k(\eta+h)\right) + \mathrm{i}(\eta_t-\gamma\eta\eta_x)\cosh(k(\eta+h))\right]\mathrm{d}x = 0, \quad (2.10a)$$

$$\xi_t + \tfrac{1}{2}\xi_x^2 + g\eta + \frac{\gamma\eta(2\eta_t\eta_x - 2\xi_x + \gamma\eta)}{2(1+\eta_x^2)} - \frac{(\eta_t+\eta_x\xi_x)^2}{2(1+\eta_x^2)} = f(\eta), \quad (2.10b)$$

where $k\in\mathbf{R}$, $\xi(x,t)=\varphi(x,\eta(x,t),t)$ *is the pseudo-potential evaluated on* $\mathcal{S}_\eta$ *and* $f(\eta)$ *is defined in* (1.4).

A similar formulation can be developed if $\mathcal{S}$ is allowed to become multivalued. Indeed, if we assume $\mathcal{S}$ is a differentiable manifold embedded in $\mathbf{R}^2$ via $x = X(\lambda, t)$ and $y = Y(\lambda, t)$, with $|Y| \to 0$ at ∞, then a calculation similar to the previous result gives:

Proposition 3 (Two Dimensional Water Waves with Constant Vorticity and Multivalued Free Surface)**.** *Let the free surface $\mathcal{S}$ be a 1-dimensional C^2-manifold, and let its embedding in $\mathbf{R}^2$ be parameterised by $X(\lambda,t)$ and $Y(\lambda,t)$. Then the boundary value problem in* (1.19a)-(1.19d) *for $(\varphi, \mathcal{S})$ is equivalent to the following pair of integro-differential equations for (ξ, X, Y):*

$$\int e^{\mathrm{i}kX}\left[\dot{\xi}\sinh\left(k(Y+h)\right) + \mathrm{i}(\dot{X}Y_t - \dot{Y}X_t - \gamma Y\dot{Y})\cosh(k(Y+h))\right]\mathrm{d}\lambda = 0, \tag{2.11a}$$

$$\xi_t + gY - \tfrac{1}{2}X_t^2 - \tfrac{1}{2}Y_t^2 + \frac{\gamma Y(2\dot{X}\dot{Y}Y_t - 2\dot{X}^2X_t - 2\dot{\xi}\dot{X} + \gamma\dot{X}^2Y)}{2(\dot{X}^2+\dot{Y}^2)} + \frac{(\dot{\xi} - \dot{X}X_t - \dot{Y}Y_t)^2}{2(\dot{X}^2+\dot{Y}^2)} - \sigma\frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2+\dot{Y}^2)^{3/2}} = 0, \tag{2.11b}$$

where $k \in \mathbf{R}$, $\xi(\lambda,t) = \varphi(X(\lambda,t), Y(\lambda,t), t)$ is the pseudo-potential evaluated on $\mathcal{S}$.

3 A Rigorous Derivation of the Linear Limit of n-Dimensional Irrotational Water Waves

Here we concentrate on the nonlinear boundary value problem described by (1.3a)-(1.3c) and (1.8) in the case $\mathcal{B}_h \equiv \mathcal{B}_0$ is constant. All derivatives are understood in the weak sense and we work with $\mathcal{S}'(\mathbf{R}^{n-1}) \supset L^2(\mathbf{R}^{n-1})$, the space of tempered distributions on $\mathbf{R}^{n-1}$. When we assert that $u \in H^s(\mathbf{R}^{n-1})$ it is to be understood in the sense that:

$$(1+\kappa^2)^{\frac{s}{2}}\hat{u}(k,\cdot) \in L^2(\mathbf{R}^{n-1}),$$

where $\hat{u} \in \mathcal{S}'(\mathbf{R}^{n-1})$ is the Fourier transform of u. The Sobolev spaces are in terms of the x-coordinates, and $u \in H^s$ means that $\|u(\cdot,t)\|_{H^s} < \infty$ for each fixed t.

Evaluating the integro-differential equations in Proposition 1 for $h = 0$, i.e $\mathcal{B}_h$ constant, we find:

$$\int e^{\mathrm{i}k\cdot x}\left[\kappa\eta_t\sinh(\kappa\eta) - \mathrm{i}k\cdot\partial_x q\cosh(\kappa\eta) + \mathrm{i}k\cdot\partial_x Q\cosh(\kappa h_0)\right]\mathrm{d}x = 0 \tag{3.1}$$

and

$$\int e^{\mathrm{i}k\cdot x}\big[\kappa\eta_t\cosh(\kappa\eta)-\mathrm{i}k\cdot\partial_x q\sinh(\kappa\eta)-\mathrm{i}k\cdot\partial_x Q\sinh(\kappa h_0)\big]\mathrm{d}x=0. \tag{3.2}$$

Now multiplying (3.1) by $\sinh(\kappa h_0)$ and (3.2) by $\cosh(\kappa h_0)$ and adding, we find the following equations:

$$\int e^{\mathrm{i}k\cdot x}\big[\kappa\eta_t\cosh(\kappa(\eta+h_0))-\mathrm{i}k\cdot\partial_x q\sinh(\kappa(\eta+h_0))\big]\mathrm{d}x=0. \tag{3.3}$$

This integro-differential equation is valid for $k\in\mathbf{R}^{n-1}$ and constitutes the global relation for the problem in (1.3a)-(1.3c) in the particular case where $\mathcal{B}_h$ is constant. Our aim is to make suitable approximations in (3.3) and to bound the relevant errors.

The linear limit is found by assuming (η,q) and certain derivatives thereof are small, in an appropriate sense, and discarding terms that are smaller. It is convenient to work on $H^2(\mathbf{R}^{n-1})\cap L^\infty(\mathbf{R}^{n-1})\subset\mathcal{S}'(\mathbf{R}^{n-1})$ with the following assumptions:

$$\|\eta\|_{H^2}<\epsilon,\qquad \|\eta_t\|_{L^1}<\epsilon,\qquad \|q_x\|_{L^2}<\epsilon,$$

for small ϵ. This assumption corresponds to solutions with small energy.

We now concentrate solely on the first term in the first integral in (3.3), since the results for the other two terms can be derived analogously.

Lemma 1. *Let $\hat\eta$ denote the Fourier transform of $\eta\in\mathcal{S}'(\mathbf{R}^{n-1})\cap L^\infty(\mathbf{R}^{n-1})$ and suppose* $\max\{\|\eta_t\|_{L^1},\|\eta\|_{L^\infty}\}<\epsilon$. *Then the following estimate holds:*

$$\int e^{\mathrm{i}k\cdot x}\frac{\eta_t\cosh(\kappa(\eta+h_0))}{\cosh(\kappa(\epsilon+h_0))}\,\mathrm{d}x=\hat\eta_t+O(\epsilon^2)$$

valid for $\kappa<O(1/\epsilon)$.

Proof. We breifly outline how to prove the following basic estimate:

$$\left|\int e^{\mathrm{i}k\cdot x}\frac{\eta_t\cosh(\kappa(\eta+h_0))}{\cosh\big(\kappa(\epsilon+h_0)\big)}\,\mathrm{d}x-\hat\eta_t\right|<2\left(1-e^{-2\kappa\epsilon}\right)\epsilon.$$

First we note that the LHS can be written as:

$$\frac{1}{\cosh\big(\kappa(\epsilon+h_0)\big)}\left|\int e^{\mathrm{i}k\cdot x}\big[\eta_t\cosh(\kappa(\eta+h_0))-\eta_t\cosh(\kappa(\epsilon+h_0))\big]\mathrm{d}x\right|. \tag{3.4}$$

Given that $\|\eta\|_{L^\infty}<\epsilon$, the following identity holds almost everywhere in $\mathbf{R}^{n-1}$:

$$\cosh\big(\kappa(\eta+h_0)\big)-\cosh\big(\kappa(\epsilon+h_0)\big)=e^{\kappa(h_0+\epsilon)}\frac{\big(1-e^{-\kappa|\eta-\epsilon|}\big)}{2}+e^{-\kappa(h_0+\epsilon)}\frac{\big(1-e^{-\kappa|\eta+\epsilon|}\big)}{2}$$

Using this in (3.4) we find:

$$\left| \int e^{\mathrm{i}k\cdot x} \frac{\eta_t \cosh(\kappa(\eta + h_0))}{\cosh\big(\kappa(\epsilon + h_0)\big)} \,\mathrm{d}x - \hat{\eta}_t \right| \le \frac{e^{\kappa(h_0+\epsilon)}}{\cosh\big(\kappa(h_0+\epsilon)\big)} \int |\eta_t| \big(1 - e^{-\kappa|\eta-\epsilon|}\big) \,\mathrm{d}x$$
$$\le 2\big(1 - e^{-2\kappa\epsilon}\big)\|\eta_t\|_{L^1}.$$

Using the a priori bound $\|\eta_t\|_{L^1} < \epsilon$, the result follows. ∎

Remark 1. We can make the bound in lemma 1 sharper by assuming that $\eta_t \in H^s(\mathbf{R}^{n-1})$ $(s > 1)$ with $\|\eta_t\|_{H^s} < \epsilon$, and then use integration by parts s times. This would improve our bound by some algebraic order in k. However, the bound is *not* uniform in κ. The asymptotic estimate in lemma 1 is valid for $\kappa < O(1/\epsilon)$, which means that the estimate is valid for sufficiently large wavelengths.

Remark 2. If we strengthen the condition on η so that $\eta \in H^s(\mathbf{R}^{n-1})$ with $s > \frac{1}{2}(n-1)$, then the standard Sobolev embedding result (see for example [11]):

$$\|\eta\|_{L^\infty} \le C\|\eta\|_{H^s}, \qquad C = C(s).$$

gives an alternative choice for the a priori bound used in the previous lemma. In this case it would be enough to assume $\|\eta\|_{H^s} < \epsilon$ for the estimates to hold, but we choose to assume the bound on $\|\eta\|_{L^\infty}$ since this appears more relevant from physical considerations.

The second term in (3.3), i.e the term

$$\int e^{\mathrm{i}k\cdot x} k \cdot \partial_x q \sinh(\kappa(\eta + h_0)) \,\mathrm{d}x,$$

can be estimated in an entirely analagous fashion with the previous result. Indeed, using a similar argument and the a priori bound $\|q_x\|_{L^1} < \epsilon$ we find:

$$\left| \int e^{\mathrm{i}k\cdot x} \left(\frac{ik\cdot\partial_x q \sinh(\kappa(\eta+h_0))}{\cosh(\kappa(\epsilon+h_0))} - \mathrm{i}k\cdot\partial_x q \tanh[\kappa h_0] \right) \mathrm{d}x \right| < 2\left(1 - e^{-2\epsilon\kappa}\right)\epsilon. \tag{3.5}$$

Then using $\partial_x \mapsto \mathrm{i}k$ in the Fourier integral, we arrive at the following result:

Lemma 2. *Let $\hat{q}$ denote the Fourier transform of $q \in \mathcal{S}'(\mathbf{R}^{n-1})$ such that $q_x \in L^1(\mathbf{R}^{n-1})$. Suppose $\max\{\|\eta\|_{L^\infty}, \|q_x\|_{L^1}\} < \epsilon$. Then the following estimate holds:*

$$\int e^{\mathrm{i}k\cdot x} \left[\frac{\mathrm{i}k\cdot\partial_x q \sinh(\kappa(\eta+h_0))}{\cosh(\kappa(h_0+\epsilon))} \right] \mathrm{d}x = -\kappa\tanh[\kappa h_0]\hat{q} + O(\epsilon^2),$$

valid for $\kappa < O(1/\epsilon)$.

The results in lemmas 1 and 2 provide us with a rigorous linear reduction of (3.3) valid for sufficiently long wave lengths, as summarised in the following.

Proposition 4. *Let* (q, η) *satisfy the boundary value problem in* (1.3a)-(1.3c) *for the case of the flat bottom* $\mathcal{B}_h = \mathcal{B}_0$. *Assume that:*

$$\max\{\|\eta_t\|_{L^1}, \|\eta\|_{L^\infty}, \|q_x\|_{L^1}\} < \epsilon.$$

Then the following estimate is valid:

$$\hat{\eta}_t - \kappa \tanh[\kappa h_0]\hat{q} = O(\epsilon^2). \tag{3.6}$$

for $\kappa < O(1/\epsilon)$.

Proof. As a consequence of the global relation (3.3), we observe the identity:

$$\begin{aligned}\hat{\eta}_t - \kappa \tanh[\kappa h_0]\hat{q} = &\left(\hat{\eta}_t - \int e^{\mathrm{i}k\cdot x} \frac{\eta_t \cosh(\kappa(\eta + h_0))}{\cosh(\kappa(h_0+\epsilon))}\,\mathrm{d}x\right) \\ &- \left(\kappa \tanh[\kappa h_0]\hat{q} - \int e^{\mathrm{i}k\cdot x}\left[\frac{\mathrm{i}k\cdot\partial_x q \sinh(\kappa(\eta+h_0))}{\cosh(\kappa(h_0+\epsilon))}\right]\mathrm{d}x\right),\end{aligned} \tag{3.7}$$

which holds for all $k \in \mathbf{R}^{n-1}$. The result follows from an application of lemmas 1 and 2. ∎

In what follows we will use the same contraints on (η, q) and linearise the Bernoulli equation *in the Fourier space*, so that we find an additional equation that couples η and q, or rather $\hat{\eta}$ and $\hat{q}$.

Recall that the Bernoulli condition reads:

$$q_t + \tfrac{1}{2}|\partial_x q|^2 + g\eta - \frac{(\eta_t + \partial_x\eta\cdot\partial_x q)^2}{2(1+|\partial_x\eta|^2)} = \frac{\sigma}{\rho}\partial_x \cdot \left(\frac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\right). \tag{3.8}$$

It is convenient to rewrite (3.8) as follows

$$q_t + g\eta - \frac{\sigma}{\rho}\Delta\eta + N(q,\eta) = 0, \tag{3.9}$$

where Δ is the standard Laplacian on $\mathbf{R}^{n-1}$ and $N(q, \eta)$ is defined as:

$$N(q,\eta) \overset{\text{def}}{=} \tfrac{1}{2}|\partial_x q|^2 + \frac{\sigma}{\rho}\partial_x\cdot\left(\frac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}} - \partial_x\eta\right) - \frac{(\eta_t + \partial_x\eta\cdot\partial_x q)^2}{2(1+|\partial_x\eta|^2)}. \tag{3.10}$$

By applying the Fourier transform to (3.9) it can be shown that under the assumption that $\|\eta\|_{H^2}$ and $\|q_x\|_{L^2}$ are sufficiently small, $\hat{N}(q,\eta)$ is negligable so that the linear terms in (3.9) are a good approximation for the dynamics. Indeed, using similar estimates to those in lemmas 1 and 2 (see appendix) we arrive at the following result.

Proposition 5. *Let* (q, η) *satisfy the Bernoulli condition* (3.8)*. Assume that:*

$$\max\{\|\eta_t\|_{L^2}, \|\eta\|_{H^2}, \|q_x\|_{L^2}\} < \epsilon.$$

Then the following estimate is valid:

$$\hat{q}_t + g\hat{\eta} + \frac{\sigma}{\rho}\kappa^2\hat{\eta} = O(\epsilon^2) \tag{3.11}$$

uniformly in κ*.*

Remark 3. The results from Propositions 4 and 5 yield the classical dispersion relation for linearised water waves. Indeed, differentiating (3.6) with respect to t and using (3.11) we find the following equation for $\hat{\eta}$:

$$\hat{\eta}_{tt} + \kappa g \tanh[\kappa h_0]\left(1 + \frac{\sigma}{g\rho}\kappa^2\right)\hat{\eta} = 0,$$

where we have discarded the $O(\epsilon^2)$ terms.

4 Formal Asymptotic Results for the two Dimensional Rotational Case

In this section we non-dimensionalise the equations in (2.10) and approach the problem perturbatively. Throughout this section we make the assumption that each of $\{\eta, \eta_t, \eta_x, \xi, \xi_x\}$ are bounded and have sufficient decay at ∞ so that the results that follow remain valid. The rigorous justification of these results should be achieved using similar arguments to those in §3, but this is not persued here. We suppose ℓ is a typical length scale for the wavelengths and a is a typical amplitude of oscillation. Then we make the following substitutions:

$$x \mapsto \ell x, \quad k \mapsto \frac{k}{\ell}, \quad t \mapsto \frac{\ell}{\sqrt{gh}}t, \quad \xi \mapsto \frac{g\ell a}{\sqrt{gh}}\xi, \quad \eta \mapsto a\eta, \quad \gamma \mapsto \frac{\sqrt{gh}}{\ell}\gamma.$$

We introduce the dimensionless parameters (ϵ, δ) defined by:

$$\epsilon = \frac{a}{h}, \qquad \delta = \frac{h}{\ell},$$

which are assumed to be small. In this case (2.10a) becomes:

$$\int e^{\mathrm{i}kx}\big\{\xi_x \sinh[\delta k(\epsilon\eta + 1)] + \mathrm{i}\delta[\eta_t - \epsilon\delta\gamma\eta\eta_x]\cosh[\delta k(\epsilon\eta + 1)]\big\}\mathrm{d}x = 0. \tag{4.1}$$

It is straightforward to individually dominate the terms appearing in the integrand, assuming appropriate bounds on $\|\xi_x\|_{L^2}$, $\|\eta\|_{H^1}$ and $\|\eta_t\|_{L^2}$. An application of the dominated convergence theorem allows us to expand the relevant

expressions as a power series in (ϵ, δ), so (4.1) yields the following expression:

$$\sum_{n,m=0}^{\infty} \epsilon^n \delta^m \int e^{ikx} A_{nm}(k, \eta, \eta_t, \xi)\, dx = 0.$$

Using the correspondence between $k \mapsto i\partial$ in the Fourier integral for a *finite* number of terms (so the relevant expression is well-defined in a classical sense) the same equation yields the following:

$$\sum_{n,m=0}^{\text{finite}} \epsilon^n \delta^m \int e^{ikx} A_{nm}(i\partial, \eta, \eta_t, \xi)\, dx + \sum_{n,m}^{\infty} \epsilon^n \delta^m \int e^{ikx} A_{nm}(k, \eta, \eta_t, \xi)\, dx = 0.$$

It is straightforward to bound the terms in the latter integral so the sum is $O(\epsilon^M \delta^N)$ for some specified M, N. Using the completeness of the Fourier transform we deduce:

$$\sum_{n,m=0}^{\text{finite}} \epsilon^n \delta^m A_{nm}(\eta, \eta_t, \xi) \sim 0,$$

where for convenience of notation we have droppped the $i\partial$ dependence. The first few A_{nm} can be easily computed:

$$A(\eta, \eta_t, \xi) = \begin{bmatrix} \eta_t + \xi_{xx} & 0 & -\frac{1}{2}\eta_{txx} - \frac{1}{6}\xi_{xxxx} & \cdots \\ (\eta\xi_x)_x & -\gamma\eta\eta_x & -(\eta\eta_t)_{xx} - \frac{1}{2}(\eta\xi_x)_{xxx} & \cdots \\ 0 & 0 & -\frac{1}{2}(\eta^2\eta_t)_{xx} - \frac{1}{2}(\eta^2\xi_x)_{xxx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{4.2}$$

We note that each of the coefficients is real, and hence (4.1) is *one* equation. This is expected, otherwise equations (2.10) would constitute an over-determined system of equations for the unknowns (η, ξ).

Now we look at the equation (2.10b) for (η, ξ). Using the non-dimensional parameters this equation becomes:

$$\begin{aligned} &\xi_t + \tfrac{1}{2}\epsilon\xi_x^2 + \eta \\ &+ \frac{\epsilon\delta\gamma\eta(2\epsilon\delta^2\eta\eta_x - 2\xi_x + \delta\eta)}{2(1 + \epsilon^2\delta^2\eta_x^2)} - \frac{\epsilon\delta^2(\eta_t + \epsilon\eta_x\xi_x)^2}{2(1 + \epsilon^2\delta^2\eta_x^2)} - \delta^2\hat{\sigma}\left(\frac{\eta_x}{\sqrt{1 + \epsilon^2\delta^2\eta_x^2}}\right)_x = 0, \end{aligned} \tag{4.3}$$

where $\hat{\sigma} \equiv \sigma/gh^2$. Again we expand the expression in terms the dimensionless parameters (ϵ, δ) to find a series of the form:

$$\xi_t + \sum_{n,m=0}^{\text{finite}} \epsilon^n \delta^m B_{nm}(\eta, \eta_t, \xi) \sim 0.$$

Again, the computation of the coefficients B_{nm} straightforward:

$$B(\eta, \eta_t, \xi) = \begin{bmatrix} \eta & 0 & -\hat{\sigma}\eta_{xx} & \cdots \\ \frac{1}{2}\xi_x^2 & -\gamma\eta\xi_x & \frac{1}{2}\gamma\eta^2 - \frac{1}{2}\eta_t^2 & \cdots \\ 0 & 0 & -\eta_t\eta_x\xi_x & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{4.4}$$

The above analysis yields the following systems of equations:

Order $\left(\epsilon^0\delta^0\right)$: To lowest order the evolution equations are:

$$\eta_t = -\xi_{xx}, \tag{4.5a}$$

$$\xi_t = -\eta. \tag{4.5b}$$

Equations (4.5) are a Hamiltonian system with the Hamiltonian density:

$$\mathcal{H}_{00}(\eta, \xi) = \frac{1}{2}\int (\eta^2 + \xi_x^2)\,\mathrm{d}x \tag{4.6}$$

and the standard symplectic structure. Indeed, (4.5) can be written in the form:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathcal{H}_{00}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.7}$$

Order $\left(\epsilon^1\delta^0\right)$: To next order the equations are:

$$\eta_t = -\xi_{xx} - \epsilon(\eta\xi_x)_x, \tag{4.8a}$$

$$\xi_t = -\eta - \tfrac{1}{2}\epsilon\xi_x^2. \tag{4.8b}$$

This system is again Hamiltonian with the following Hamiltonian density:

$$\mathcal{H}_{10}(\eta, \xi) = \mathcal{H}_{00}(\eta, \xi) + \epsilon\frac{1}{2}\int \eta\xi_x^2\,\mathrm{d}x. \tag{4.9}$$

Indeed (4.8) can be written in the form:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathcal{H}_{10}. \tag{4.10}$$

Order $\left(\epsilon^1\delta^1\right)$: To this order the evolution equations are:

$$\eta_t = -\xi_{xx} - \epsilon(\eta\xi_x)_x + \epsilon\delta\gamma\eta\eta_x, \tag{4.11a}$$

$$\xi_t = -\eta - \tfrac{1}{2}\epsilon\xi_x^2 + \epsilon\delta\gamma\eta\xi_x. \tag{4.11b}$$

The associated Hamiltonian density is:

$$\mathcal{H}_{11}(\eta, \xi) = \mathcal{H}_{10}(\eta, \xi) + \epsilon\delta\int \gamma\eta\eta_x\xi\,\mathrm{d}x. \tag{4.12}$$

Indeed, (4.11) can be written in the form:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathcal{H}_{11}. \tag{4.13}$$

Order $\left(\epsilon^0\delta^2\right)$: At this order these exists a slight complication, because the RHS of the evolution equations involves η_t:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathcal{H}_{11} + \delta^2 \begin{pmatrix} \frac{1}{2}\eta_{txx} + \frac{1}{6}\xi_{xxxx} \\ \hat{\sigma}\eta_{xx} \end{pmatrix}. \tag{4.14}$$

However, by using the expression for η_t recursively, we can express the RHS in terms of (η, ξ) and x-derivatives thereof. One must keep track of the order of the relevant terms in the recursive routine. Implementing this approach gives to $O(\delta^2)$ the following equations:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathcal{H}_{11} + \delta^2 \begin{pmatrix} -\frac{1}{3}\xi_{xxxx} \\ \hat{\sigma}\eta_{xx} \end{pmatrix}. \tag{4.15}$$

This system is again Hamiltonian with respect to the Hamiltonian density given by

$$\mathcal{H}_{02} = \mathcal{H}_{11} + \tfrac{1}{2}\delta^2 \int (\hat{\sigma}\eta_x^2 - \tfrac{1}{3}\xi_{xx}^2)\,\mathrm{d}x. \tag{4.16}$$

Order $\left(\epsilon^1\delta^2\right)$: Again computing the relevant $O(\epsilon\delta^2)$ terms through the recursion process, we find the following Hamiltonian system:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathcal{H}_{02} + \epsilon\delta^2 \begin{pmatrix} -(\eta\xi_{xx})_{xx} \\ \frac{1}{2}\xi_{xx}^2 - \gamma^2\eta^2 \end{pmatrix}. \tag{4.17}$$

The associated Hamiltonian density is:

$$\mathcal{H}_{21} = \mathcal{H}_{20} + \tfrac{1}{2}\epsilon\delta^2 \int (\tfrac{1}{3}\gamma^2\eta^3 - \xi_{xx}^2\eta)\,\mathrm{d}x. \tag{4.18}$$

The above analysis suggests that the system is Hamitlonian at all orders, with the infinite chain of Hamiltonians:

$$\begin{tikzcd}
\mathcal{H}_{00} \arrow[d] & \mathcal{H}_{01} \arrow[r] & \mathcal{H}_{02} \arrow[dl] & \mathcal{H}_{03} \\
\mathcal{H}_{10} \arrow[ur] & \mathcal{H}_{11} \arrow[dl] & \mathcal{H}_{12} \arrow[ur] & \\
\mathcal{H}_{20} \arrow[d] & \mathcal{H}_{21} \arrow[ur] & & \\
\mathcal{H}_{30} \arrow[ur] & & &
\end{tikzcd}$$

Conjecture. *The full system is Hamiltonian with respect to the following Hamiltonian:*

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} \mathcal{H}_{nm}$$

where the grading is with respect to the perturbation parameters (ϵ, δ).

Eliminating η from equations (4.15) and discarding terms of order $O(\epsilon^N \delta^M)$ with $N + M > 2$ we find the equation:

$$\xi_{tt} - \xi_{xx} + \epsilon(2\xi_x\xi_{xt} + \xi_t\xi_{xx}) \\ + \epsilon\delta\gamma(2\xi_t\xi_{xt} + \xi_x\xi_{tt}) + \tfrac{1}{3}\epsilon^2\xi_x^2\xi_{xx} + \delta^2\left(\hat{\sigma} - \tfrac{1}{3}\right)\xi_{xxxx} = 0. \tag{4.19}$$

We introduce the long length scale X and the fast time scale T by the equations

$$\epsilon X = x, \qquad \epsilon T = t.$$

These equations imply $\partial_x = \epsilon^{-1}\partial_X$ and $\partial_t = \epsilon^{-1}\partial_T$ and (4.19) becomes:

$$\xi_{TT} - \xi_{XX} + (2\xi_X\xi_{XT} + \xi_T\xi_{XX}) \\ + \delta\gamma(2\xi_T\xi_{XT} + \xi_X\xi_{TT}) + \tfrac{1}{3}\xi_X^2\xi_{XX} + \epsilon^{-2}\delta^2\left(\hat{\sigma} - \tfrac{1}{3}\right)\xi_{XXXX} = 0. \tag{4.20}$$

Now suppose the vorticity is large, namely $\gamma = O(\delta^{-1})$ and set $\kappa = \gamma\delta$ which is then $O(1)$. In addition, set $\delta = \epsilon$. Then (4.20) becomes:

$$\xi_{TT} - \xi_{XX} + 2\xi_X\xi_{XT} + \xi_T\xi_{XX} \\ + \kappa(2\xi_T\xi_{XT} + \xi_X\xi_{TT}) + \tfrac{1}{3}\xi_X^2\xi_{XX} + \left(\hat{\sigma} - \tfrac{1}{3}\right)\xi_{XXXX} = 0. \tag{4.21}$$

This equation is *different* to the analagous equation for irrotational waves. Indeed, performing calculations similar to those by [1], one finds

$$q_{TT} - q_{XX} + 2q_Xq_{XT} + q_Tq_{XX} + \tfrac{1}{3}q_X^2q_{XX} + \left(\hat{\sigma} - \tfrac{1}{3}\right)q_{XXXX} = 0,$$

where q is the velocity potential on the free surface of the *irrotational* fluid.

We note that if we use in (4.19) the change of variables

$$\chi = x - t, \qquad \tau = \epsilon t,$$

and also set $\epsilon = \delta^2$, then to leading order equation (4.19) gives

$$\left(2\xi_\tau + \xi_\chi^2 + \left(\hat{\sigma} - \tfrac{1}{3}\right)\xi_{\chi\chi\chi}\right)_\chi = 0, \tag{4.22}$$

which is in agreement with a similar result derived by [1].

Proposition 6. *Equation* (4.21) *admits two 1-parameter families of solitons under the following constraints:*

1. *$\hat{\sigma} > \frac{1}{3}$, $c < 1$ and $\kappa = O(1)$ arbitrary.*

2. *$\hat{\sigma} < \frac{1}{3}$, $c > 1$ and either:*

$$(i)\ \ \kappa < \frac{1}{c} - \frac{\sqrt{2}}{c}\sqrt{1 - \frac{1}{c^2}} \quad \text{or} \quad (ii)\ \ \kappa > \frac{1}{c} + \frac{\sqrt{2}}{c}\sqrt{1 - \frac{1}{c^2}}.$$

The cases (i) and (ii) correspond to elevated and depression solitons respectively.

Proof. We consider travelling wave solutions in the form $\xi(X,T) = f(z)$, where $z = X - cT$. Using this ansatz in (4.21) and applying the chain rule gives the ordinary differential equation:

$$(c^2 - 1)f'' + 3c(\kappa c - 1)f'f'' + \tfrac{1}{3}f''(f')^2 + \left(\hat{\sigma} - \tfrac{1}{3}\right)f^{(4)} = 0, \tag{4.23}$$

where the prime denotes differentiation with respect to z. We can integrate (4.23) with respect to z to give:

$$(c^2 - 1)f' + \tfrac{3c}{2}(\kappa c - 1)(f')^2 + (f')^3 + \left(\hat{\sigma} - \tfrac{1}{3}\right)f^{(3)} = 0, \tag{4.24}$$

where we have assumed the derivatives of f vanish as $|z| \to \infty$. Now multiplying (4.24) by f'' and integrating once more gives

$$\alpha(f')^2 + \beta(f')^3 - \mu(f')^4 - (f'')^2 = 0, \tag{4.25}$$

where we have discarded constants of integration and introduced the constants:

$$\alpha' = \frac{1 - c^2}{\left(\hat{\sigma} - \frac{1}{3}\right)}, \qquad \beta = \frac{c(1 - \kappa c)}{\left(\hat{\sigma} - \frac{1}{3}\right)}, \qquad \mu = \frac{1}{2\left(\hat{\sigma} - \frac{1}{3}\right)}.$$

For real solutions we require $\alpha' > 0$, so we introduce set $\alpha' = \alpha^2$. There are two seperate cases in which this condition holds: (i) $|c| < 1$ and $\hat{\sigma} > \frac{1}{3}$ or (ii) $|c| > 1$ and $\hat{\sigma} < \frac{1}{3}$. Now we introduce $w(z) = f'(z)$, the physical variable corresponding to the speed of the wave propagating along $\mathcal{S}_\eta$. Rewriting (4.25) as an ODE in w:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^2 = \frac{1}{w^2(\alpha^2 + \beta w - \mu w^2)}. \tag{4.26}$$

At this point it is useful to observe some obvious symmetries in (4.26):

$$\begin{aligned}
G_1 &: (w, z; \alpha, \beta, \mu) \mapsto (-w, z; \alpha, -\beta, \mu),\\
G_2 &: (w, z; \alpha, \beta, \mu) \mapsto (w, z; -\alpha, \beta, \mu),\\
G_3 &: (w, z; \alpha, \beta, \mu) \mapsto (w, z + z_0; \alpha, \beta, \mu),\\
G_4 &: (w, z; \alpha, \beta, \mu) \mapsto (w, -z; \alpha, \beta, \mu)
\end{aligned}$$

In particular, given a solution $w = f(z;\alpha,\beta,\mu)$, these symmetries generate the new solutions:

$$\begin{aligned}(G_1 \circ f)(z;\alpha,\beta,\mu) &= -f(z,\alpha,-\beta,\mu),\\ (G_2 \circ f)(z;\alpha,\beta,\mu) &= f(z,-\alpha,\beta,\mu),\\ (G_3 \circ f)(z;\alpha,\beta,\mu) &= f(z-z_0,\alpha,\beta,\mu),\\ (G_4 \circ f)(z;\alpha,\beta,\mu) &= f(-z,\alpha,\beta,\mu).\end{aligned}$$

This will prove useful in what follows. Integrating equation (4.26) and invoking each of the above symmetries gives the solutions:

$$w(\pm z + z_0) = \frac{\pm 4\alpha^3 \exp(\alpha z)}{4\alpha^2\mu + \beta^2 \mp 2\alpha\beta\exp(\alpha z) + \alpha^2\exp(2\alpha z)}$$

where z_0 is constant. It is clear that the actions of G_1 and $G_2 \circ G_4$ are equivalent, so we are left with four different 1-parameter families of solutions. However, some algebra shows that the action of G_4 is equivalent to the horizontal translation:

$$z \mapsto z + \frac{1}{\alpha}\log\left(4\mu + \frac{\beta^2}{\alpha^2}\right),$$

so the symmetries $\{G_i\}$ only generate two independent, 1-parameter families of solutions:

$$w_\uparrow(z+z_0) = \frac{4\alpha^3\exp(\alpha z)}{4\alpha^2\mu + \beta^2 - 2\alpha\beta\exp(\alpha z) + \alpha^2\exp(2\alpha z)}, \tag{4.27a}$$

$$w_\downarrow(z+z_0) = \frac{-4\alpha^3\exp(\alpha z)}{4\alpha^2\mu + \beta^2 + 2\alpha\beta\exp(\alpha z) + \alpha^2\exp(2\alpha z)}, \tag{4.27b}$$

where the arrows indicate that the wave is elevated (classical soliton), or a depression wave. Observe that in each case the denominator is of the form:

$$[\alpha\exp(\pm\alpha z) \mp \beta]^2 + 4\alpha^2\mu.$$

It is clear that in case (i) $|c| < 1$ and $\hat{\sigma} > \frac{1}{3}$, this expression is positive for all z, since $\mu > 0$. So for $|c| < 1$ and $\hat{\sigma} > \frac{1}{3}$, equation (4.21) admits *both* depression solitons and elevated solitons.

In case (ii) we see that $\mu < 0$, so equations (4.27) do not give soliton solutions when the denominator is allowed to vanish. For $w_\uparrow$ to give a soliton solution, we require $\beta < 0$ and $4\alpha^2\mu + \beta^2 > 0$. Using the definitions of α, β, μ, we see this is equivalent to:

$$\begin{aligned}c(1-\kappa c) &> 0,\\ c^2(1-\kappa c)^2 + 2(1-c^2) &> 0.\end{aligned}$$

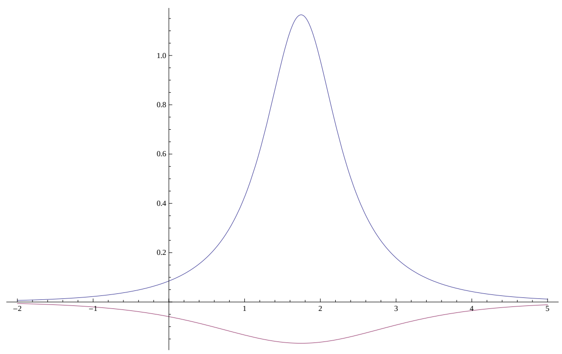

Figure 1: Depression and elevated solitons corresponding to $(c, \kappa, \hat{\sigma}) = (0.95, 0.5, 0.4)$.

We take $c > 1$ and so the first inequality gives $\kappa c < 1$, and the second gives:

$$\kappa < \frac{1}{c} - \frac{\sqrt{2}}{c}\sqrt{1 - \frac{1}{c^2}}. \tag{4.28}$$

Similarly, for $w_{\downarrow}$ to give a soliton solution we require $\beta > 0$ and $4\alpha^2\mu + \beta^2 > 0$. In analogy with (4.28) we find the condtion:

$$\kappa > \frac{1}{c} + \frac{\sqrt{2}}{c}\sqrt{1 - \frac{1}{c^2}}. \tag{4.29}$$

This completes the proof of the result in Proposition 6 ■

We note that in the absense of vorticity (i.e $\kappa = 0$), neither of the conditions (4.28) and (4.29) can be satisfied, so the equation (4.21) admits no classical soliton solutions when $\hat{\sigma} < \frac{1}{3}$. This is in agreement with the rigorous non-existence results of [13].

It should be noted that the PDE in (4.21) is not well-posed in general when $\hat{\sigma} < \frac{1}{3}$. Considering plane wave perturbations with small amplitude leads to the dispersion relation:

$$\omega^2 = k^2 + \left(\hat{\sigma} - \tfrac{1}{3}\right) k^4.$$

In the case $\hat{\sigma} < \frac{1}{3}$ we see that the amplitude of the perturbations will grow exponentially in time. This indicates that solitons derived in case (b) of Proposition 6 will be highly unstable. Nevertheless, it is interesting to look at the behaviour of these solitons which are completely ruled out in the absense of vorticity. It is convenient to perform a translation in z so that the solutions are symmetric

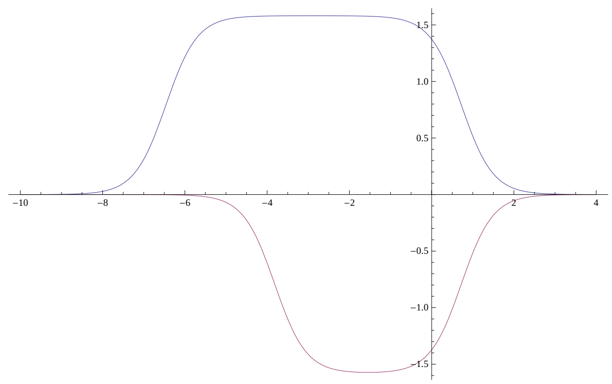

Figure 2: Depression and elevated solitons in the case $\hat{\sigma} < \frac{1}{3}$ for $\kappa = \frac{3}{2}$ and $c > 1$.

about $z = 0$. Some algebra gives:

$$w_{\uparrow}(z) = \frac{\Gamma_1}{1 + \Gamma_2 \cosh(\alpha z)}$$
$$w_{\downarrow}(z) = \frac{\Gamma_1}{1 - \Gamma_2 \cosh(\alpha z)}$$

where we have introduced the two constants:

$$\Gamma_1 \equiv -\frac{2\alpha^2}{\beta}, \qquad \Gamma_2 \equiv \frac{1}{\beta}\sqrt{4\mu\alpha^2 + \beta^2}.$$

In fact, if we don't specify which side of the branch we are on in the $(\cdot)^{1/2}$ term in Γ_2, these two solutions can be encorperated into one single expression:

$$W(z) = \frac{\Gamma_1}{1 + \Gamma_2 \cosh(\alpha z)}, \tag{4.30}$$

which then describes elevated solitons when we take the positive square root and describes depression solitons when we take the negative square root. In this case the constants $\{\Gamma_1, \Gamma_2\}$ are determined by the points on a 1-dimensional manifold Γ:

$$\Gamma = \{(\Gamma_1, \Gamma_2) \in \mathbf{R}^2 : \Gamma_2^2 - \left(\frac{\mu}{\alpha^2}\right)\Gamma_1^2 = 1\} \tag{4.31}$$

In summary, the soliton solutions of equation (4.21) are given by $\xi(X, T) = W(X - cT)$ where:

$$W(z) = \frac{\Gamma_1}{1 + \Gamma_2 \cosh(\alpha z)}, \quad (\Gamma_1, \Gamma_2) \in \Gamma. \tag{4.32}$$

There is now a clear geometrical interpretation of the different types of soliton solution to (4.21) in terms of the curve Γ. The topology of Γ is completely

determined by the sign of μ/α^2. Using the definitions of α and μ, one sees this is equivalent to the sign of $\hat{\sigma} - \frac{1}{3}$, or the sign of $c^2 - 1$.

1. $\left(\hat{\sigma} > \frac{1}{3}\right)$. In this case Γ is hyperbola, i.e. two disconnected copies of $\mathbf{R}$. There is a bifurcation on each of these disconnected componets at $\Gamma_2 = 0$, which corresponds to constant solution $W = \Gamma_1$. Removing these two points we are left with four disconnected components. The two components with $\Gamma_1\Gamma_2 > 0$ correspond to elevated solitons, whereas the two components with $\Gamma_1\Gamma_2 < 0$ correspond to depression solitons.

2. $\left(\hat{\sigma} < \frac{1}{3}\right)$. In this case $\Gamma \simeq S^1$. In this case there are four bifurcations. As in the previous case, two of these bifurcations are at $\Gamma_2 = 0$ and correspond to the constant solution $W = \Gamma_1$. However, at the bifurcations points $\Gamma_1 = 0$, the solution becomes unphysical and has finite time blow-up. Removing the points of bifurcation, we are left with four disconnected components. The two components in $\Gamma_2 > 0$ correspond to elevated solitons ($\Gamma_1 > 0$) and depression solitons ($\Gamma_1 < 0$). The remaining two disconnected components correspond to unphysical solutions which blow up at $\alpha z = \operatorname{arccosh}(|\Gamma_2|^{-1})$.

5 Conclusions

We have presented the following generalisations of the results of [1]: (a) We have extended the formalism to the case of n-dimensions, $n > 1$; (b) we have considered the case of constant vorticity; and (c) we have incorperated the effect of a multi-valued free surface.

We recall that [7] introduced an elegant Dirichlet to Neumann operator $G(\eta)$ associated with the velocity potential and also obtained a series for the operator $G(\eta)$, valid for small η. The integral equations present in [1] and here, can be considered as the *summation* of the above series, i.e. the series of [7] is the Neumann series of the integral equations derived by [1] and here.

It appears that the new formulation provides an alternative, perhaps simpler approach, for (a) the numerical investigation of water waves; (b) the derivation of various asymptotic limits; (c) the rigorous analysis of water waves.

Regarding (a), we recall that two-dimensional lumps were computed by [1] in the case of sufficient surface tension.

Regarding (b) we recall that various asymptotic equations, including the Boussinesq, Benney-Luke and the nonlinear Schrödinger equations, were derived by [1]. Similarly, an appropriate Boussinesq type equation in the case of constant vorticity is found in §4. We observe that these equations have been derived by several other authors, however, it appears that the new formulation

provides a straightforward way of deriving these equations. For example, regarding our results in §4, we note that [5] uses a perturbative approach which involves solving many PDEs arising from the consistency of the perturbation expansion with the boundary conditions. To solve these PDEs, the author confines attention to seperable solutions. The entire paper is devoted to computing the first few coefficients of the perturbation expansion. The author comments that this method is unweildy at anything beyond second order. Our paper gives a much more direct approach, and can easily be extended to higher orders. It is not immediately obvious as to whether his equations are exactly the same as ours, because our soliton equation is for the wave speed, whilst his are for the wave height.

Regarding (c) we have shown in §3 that standard PDE techniques can be used for the rigorous analysis of water waves, at least in the linear limit. The new formulation suggests a rigorous methodology which differs drastically from that employed in the important works of [2, 6, 16]; the extension of the results of §3 to the nonlinear problem is a work in progress.

Finally, we note that in the classical works of [4], the author focuses entirely on the amplitude $\eta(x,t)$ and does not include the effect of surface tension. In contrast, the solitons discussed in this work concern the wave speed, $\xi(x,t)$, and our formulation also includes the effect of surface tension. Our analysis, particularly the results in Proposition 6, outline the importance of the relationship between the surface tension and the vorticity for the existence of solitary waves. Furthermore, our formalism can easily be extended to the three dimensional case. The main advantages of the new approach presented by [1] and in this paper, are a consequence of the *explicit* nature of the equations in Propositions 1, 2 and 3.

6 Appendix

Here we prove the result in Proposition 5. It suffices to prove the estimate:

$$\|N(\eta, q)\|_{L^1} = O(\epsilon^2)$$

given that $\max\{\|\eta\|_{H^2}, \|q\|_{H^1}\|\} < \epsilon$. Recall that:

$$N(q,\eta) \stackrel{\text{def}}{=} \tfrac{1}{2}|\partial_x q|^2 + \frac{\sigma}{\rho}\partial_x \cdot \left(\frac{\partial_x \eta}{\sqrt{1+|\partial_x \eta|^2}} - \partial_x \eta\right) - \frac{(\eta_t + \partial_x \eta \cdot \partial_x q)^2}{2(1+|\partial_x \eta|^2)}.$$

The following estimate is clear from the definition:

$$\|e^{ik\cdot x} N(\eta,q)\|_{L^1} \leq \tfrac{1}{2}\|\partial_x q\|_{L^2}^2 + \tfrac{\sigma}{\rho}\left\|\partial_x \cdot \left(\tfrac{\partial_x \eta}{\sqrt{1+|\partial_x \eta|^2}} - \partial_x \eta\right)\right\|_{L^1} + \tfrac{1}{2}\left\|\tfrac{\eta_t + \partial_x \eta \cdot \partial_x q}{\sqrt{1+|\partial_x \eta|^2}}\right\|_{L^2}^2.$$

The first term is clearly $O(\epsilon^2)$ since $\|q\|_{H^1} < \epsilon$. For the third term, we use the estimate:

$$\begin{aligned}\left\|\tfrac{\eta_t + \partial_x \eta \cdot \partial_x q}{\sqrt{1+|\partial_x \eta|^2}}\right\|_{L^2}^2 &\leq \left(\|\eta_t\|_{L^2} + \left\||\partial_x q|\left[\tfrac{|\partial_x \eta|}{\sqrt{1+|\partial_x \eta|^2}}\right]\right\|_{L^2}\right)^2 \\ &\leq \left(\|\eta_t\|_{L^2} + \|\partial_x q\|_{L^2}\right)^2 \qquad (6.1)\end{aligned}$$

which gives the required $O(\epsilon^2)$ bound. For the second term, note that:

$$\left\|\partial_x \cdot \left(\frac{\partial_x \eta}{\sqrt{1+|\partial_x \eta|^2}} - \partial_x \eta\right)\right\|_{L^1} \leq \left\|\left(\frac{1}{\sqrt{1+|\partial_x \eta|^2}} - 1\right)\Delta\eta\right\|_{L^1} + \sum_{i,j}\left\|\frac{\eta_i \eta_j \eta_{ij}}{(1+|\partial_x \eta|^2)^{3/2}}\right\|_{L^1}$$

where $\eta_i \equiv \partial\eta/\partial x_i$. Rewriting the second term, we observe the following estimate:

$$\begin{aligned}\sum_{i,j}\left\|\frac{\eta_i \eta_j \eta_{ij}}{(1+|\partial_x \eta|^2)^{3/2}}\right\|_{L^1} &= \sum_{i,j}\left\|\left(\frac{\eta_i}{(1+\sum_k \eta_k^2)^{3/2}}\right)\eta_{ij}\eta_j\right\|_{L^1} \\ &\leq \sum_{i,j}\left\|\left(\frac{\eta_i}{(1+\eta_i^2)^{3/2}}\right)\eta_{ij}\eta_j\right\|_{L^1} \\ &\leq \sum_{i,j}\tfrac{2}{3\sqrt{3}}\|\eta_{ij}\eta_j\|_{L^1} \\ &\leq \sum_{i,j}\tfrac{2}{3\sqrt{3}}\|\eta_{ij}\|_{L^2}\|\eta_j\|_{L^2}, \qquad (6.2)\end{aligned}$$

where we applied the Cauchy-Schwarz inequality. Similarly, we find:

$$\left\|\left(\frac{1}{\sqrt{1+|\partial_x \eta|^2}} - 1\right)\Delta\eta\right\|_{L^1} \leq \tfrac{1}{2}\|\partial_x \eta\|_{L^2}\|\Delta\eta\|_{L^2}. \qquad (6.3)$$

Combining (6.2) and (6.3) we find:

$$\left\| \partial_x \cdot \left(\frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}} - \partial_x \eta \right) \right\|_{L^1} \le \|\eta\|_{H^2}^2. \tag{6.4}$$

From the estimates in (6.1) and (6.4) it is follows that $\|N(\eta, q)\|_{L^1} = O(\epsilon^2)$.

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